Note on equivalent Lagrangians and symmetries

1983 J. Phys. A: Math. Gen. 16 L229

(http://iopscience.iop.org/0305-4470/16/7/006)

J. Phys. A: Math. Gen. 16 (1983) L229–L233. Printed in Great Britain

LETTER TO THE EDITOR

Note on equivalent Lagrangians and symmetries

Willy Sarlet

Instituut voor Theoretische Mechanica, Rijksuniversiteit Gent, Krijgslaan 281, B-9000 Gent, Belgium

Received 14 January 1983

Abstract. We show that every ordered pair of equivalent Lagrangians determines an equivalence class of dynamical symmetries of non-Noether type. Its explicit construction merely requires a particular solution of a single partial differential equation. Various related aspects are discussed.

Many recent papers deal with aspects of the study of symmetries and first integrals for second-order ordinary differential equations. For Lagrangian systems, the best known type of symmetries are of course the Noether symmetries, which directly provide a corresponding first integral, but a number of interesting features have been revealed concerning non-Noether symmetries too. For a few of these, one can consult e.g. Lutzky (1979, 1982a, b), Crampin (1980), González-Gascón and Rodriguez-Camino (1980a, b, c), Steeb (1982), Sarlet and Cantrijn (1981b). One specific property which was observed by many authors is this: when a point symmetry of a Lagrangian system is of non-Noether type, it leads to an equivalent Lagrangian (or sometimes only a 'subordinate' one). See in this respect Marmo and Saletan (1977), Lutzky (1978) and Prince (1983). Prince made an attempt to generalise this property for other than point symmetries. So far, however, one is led to believe that for velocity-dependent symmetries no general rule exists by which non-Noether symmetries could be associated to equivalent Lagrangians. But what about the converse? Whether a given pair of equivalent Lagrangians can lead to a dynamical symmetry was also termed an open question by Prince. We wish to show here that this question at least has a simple answer: any two equivalent Lagrangians yield two corresponding non-Noether symmetries in a quite unique way. Their construction, moreover, is relatively simple and merely requires a particular solution of a single partial differential equation.

Consider a given second-order vector field

$$\Gamma = \partial/\partial t + \dot{q}^i\, \partial/\partial q^i + \Lambda^i(t, q, \dot{q})\, \partial/\partial \dot{q}^i, \tag{1}$$

which is supposed to be derivable from a regular Lagrangian L, that is, we have $i_\Gamma \mathrm{d}\theta(L) = 0$, where $\theta(L)$ is the Cartan form

$$\theta(L) = L\, \mathrm{d}t + (\partial L/\partial \dot{q}^i)(\mathrm{d}q^i - \dot{q}^i\, \mathrm{d}t). \tag{2}$$

A dynamical symmetry of (1) is any vector field $Y = \tau\partial/\partial t + \xi^i\partial/\partial q^i + \eta^i\partial/\partial \dot{q}^i$ whose Lie bracket with Γ yields a multiple of Γ. As is customary when dealing with Lagrangian systems, we define two dynamical symmetries to be equivalent when they differ by a multiple of Γ (see e.g. Sarlet and Cantrijn 1981a). Each equivalence class of symmetries then contains a representative $\bar{Y} = \mu^i \partial/\partial q^i + \nu^i \partial/\partial \dot{q}^i$, which is a genuine symmetry of Γ in the sense that $[\bar{Y}, \Gamma] = 0$, or equivalently

$$\nu^i = \Gamma(\mu^i), \qquad \Gamma(\nu^i) = \bar{Y}(\Lambda^i). \tag{3}$$

The connection between Y and $\bar{Y}$ is determined by

$$\mu^i = \xi^i - \dot{q}^i \tau, \qquad \nu^i = \eta^i - \Lambda^i \tau. \tag{4}$$

Y is called a point symmetry when τ and ξ^i do not depend on the $\dot{q}$. Y is a Noether symmetry with respect to L when $\mathscr{L}_Y \mathrm{d}\theta(L) = 0$. The key relation in Prince's attempt to associate an equivalent Lagrangian with a given non-Noether symmetry Y is

$$\mathscr{L}_Y \theta(L) = \theta(L') + \mathrm{d}f. \tag{5}$$

In the present context, we study this relation under the assumption that an equivalent (regular) Lagrangian L' is given and Y is to be determined.

Lemma 1. If Y satisfies (5) and (L, L') are equivalent Lagrangians, then Y is a dynamical symmetry.

Proof. From $i_\Gamma \mathrm{d}\theta(L) = i_\Gamma \mathrm{d}\theta(L') = 0$ and $\mathscr{L}_Y\, \mathrm{d}\theta(L) = \mathrm{d}\theta(L')$ it follows that

$$i_{[\Gamma, Y]}\, \mathrm{d}\theta(L) = i_\Gamma \mathscr{L}_Y\, \mathrm{d}\theta(L) - \mathscr{L}_Y i_\Gamma\, \mathrm{d}\theta(L) = 0,$$

which in view of the regularity of L implies that $[Y, \Gamma]$ is a multiple of Γ.

We are now left with the question whether a Y satisfying (5) exists and how to construct it. To this end it is worthwhile recognising first the following necessary conditions.

Lemma 2. If Y satisfies (5), we have

$$(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j)(\xi^j - \dot{q}^j \tau) = -\partial G/\partial \dot{q}^i, \qquad \Gamma(G) = -L', \tag{6a,b}$$

with $G = f - \langle Y, \theta(L) \rangle$. (7)

Proof. Equation (5) can trivially be rewritten as

$$i_Y\, \mathrm{d}\theta(L) = \theta(L') + \mathrm{d}G, \tag{8}$$

with G as in (7). Taking the inner product of (8) with $\partial/\partial \dot{q}^i$ and Γ respectively immediately yields (6).

The main observation to make now is that finding a particular solution G of the single partial differential equation (6b) is sufficient to construct a dynamical symmetry Y satisfying (5).

Theorem 1. Let L' be an equivalent Lagrangian and G be an arbitrary particular solution of (6b); define $\mu^i = \xi^i - \dot{q}^i \tau$ by the algebraic relations (6a) and set

$$\eta^i = \Gamma(\xi^i) - \dot{q}^i \Gamma(\tau) \qquad (\text{or } \nu^i = \Gamma(\mu^i)). \tag{9}$$

Then Y (or $\bar{Y}$) defines an equivalence class of symmetries associated with the pair (L, L') through (5).

Proof. Contracting $\mathrm{d}\theta(L)$ with vector fields defines a mapping from vector fields into one-forms, the kernel of which is one dimensional. The condition (6*b*) guarantees that $\theta(L')+\mathrm{d}G$ lies in the range of that mapping (see e.g. Crampin 1977). Therefore, there exists a vector field $\tilde{Y}(\tilde{\tau}, \tilde{\xi}^i, \tilde{\eta}^i)$ satisfying

$$i_{\tilde{Y}}\, \mathrm{d}\theta(L)=\theta(L')+\mathrm{d}G. \tag{10}$$

From (10) and lemma 2, it is clear that $(\tilde{\tau}, \tilde{\xi}^i)$ satisfy (6*a*) too, hence we have

$$\tilde{\xi}^i-\dot{q}^i\tilde{\tau}=\xi^i-\dot{q}^i\tau. \tag{11}$$

In addition, (10) implies that $\tilde{Y}$ satisfies a relation of type (5) and therefore is a dynamical symmetry. This in particular implies (according to the conditions (3) and (4)) that

$$\tilde{\eta}^i=\Gamma(\tilde{\xi}^i)-\dot{q}^i\Gamma(\tilde{\tau}). \tag{12}$$

Defining a function h by $h=\tilde{\tau}-\tau$, the relations (9), (11) and (12) straightforwardly yield $Y=\tilde{Y}-h\Gamma$, from which the result follows.

Concerning the degree of arbitrariness in determining Y, we note that if G_1 is a particular solution of (6*b*), any other solution G_2 differs from G_1 by a constant of the motion F. If μ_1^i and μ_2^i denote the corresponding symmetry components, determined through (6*a*), their difference μ^i will satisfy

$$(\partial^2L/\partial\dot{q}^i\partial\dot{q}^j)\,\mu^j=-\partial F/\partial\dot{q}^i, \qquad \Gamma(F)=0, \tag{13}$$

which is sufficient to conclude that we are dealing with a Noether symmetry with respect to L (Sarlet and Cantrijn 1981a). We therefore can state:

Corollary 1. Apart from the previously discussed equivalence relation, the non-Noether symmetry Y corresponding to the ordered pair (L, L') is unique up to an arbitrary Noether-symmetry with respect to L.

Obviously, we can interchange the role of L and L' in everything above. In other words, we can search for a particular solution G' of the equation

$$\Gamma(G')=-L. \tag{14}$$

The procedure of theorem 1 then determines another non-Noether symmetry Y', corresponding to the ordered pair (L', L). Now, contraction of a Cartan form with two dynamical symmetries always produces a first integral. Many authors have discussed means of associating first integrals with a couple of equivalent Lagrangians (see some of the earlier citations and also Hojman and Harleston (1981)). In the present context, we here identify a different and rather natural procedure for associating two first integrals with the pair (L, L').

Corollary 2. To each pair of equivalent Lagrangians L and L' correspond two first integrals F and F', defined by

$$i_{Y'}i_Y\,\mathrm{d}\theta(L)=F, \qquad i_Yi_{Y'}\,\mathrm{d}\theta(L')=F'. \tag{15}$$

In terms of G and G', we have

$$F=Y'(G)+\langle Y, \theta(L')\rangle, \qquad F'=Y(G')+\langle Y, \theta(L)\rangle. \tag{16}$$

Some final remarks are in order now. In general, finding a dynamical symmetry Y of a second-order vector field Γ requires, in accordance with (3), solving n partial differential equations of second order. We have seen here that when two equivalent Lagrangians are known, a symmetry can be obtained by solving a single partial differential equation of first order. This is of course not the only instance of that sort. Indeed, a similar statement certainly holds when only one Lagrangian is known via Noether's theorem, the partial differential equation then being the Liouville equation $\Gamma(F)=0$. But there is more! Indeed, it is clear that the whole procedure of theorem 1 remains valid when we let L and L' coincide. We are then in fact talking about a conformal symmetry of the two-form $\mathrm{d}\theta$ and as such are falling back on a particular case of the theory developed by González-Gascón and Rodriguez-Camino (1980a, b). Explicitly, we could take the function G' satisfying (14) and replace G by G' in the right-hand side of (6a) to obtain a dynamical symmetry X, satisfying $\mathscr{L}_X \mathrm{d}\theta(L)=\mathrm{d}\theta(L)$. For two equivalent Lagrangians, our functions G and G' therefore determine four symmetries Y, Y', X, X', which of course need not be independent.

As an illustrative example, consider the equivalent Lagrangians

$$L=\tfrac{1}{2}(\dot q_1^2-\dot q_2^2)-\tfrac{1}{2}a(q_1^2-q_2^2)-bq_1q_2, \tag{17}$$

$$L'=\dot q_1\dot q_2-aq_1q_2-\tfrac{1}{2}b(q_2^2-q_1^2), \tag{18}$$

where a and b are constants. Equation (6b) has the particular solution

$$G=-\tfrac{1}{2}(q_1\dot q_2+q_2\dot q_1), \tag{19}$$

while

$$G'=\tfrac{1}{2}(q_2\dot q_2-q_1\dot q_1) \tag{20}$$

solves (14). The corresponding symmetries are given by

$$Y=\tfrac{1}{2}(q_2\,\partial/\partial q_1-q_1\,\partial/\partial q_2)+\tfrac{1}{2}(\dot q_2\,\partial/\partial \dot q_1-\dot q_1\,\partial/\partial \dot q_2), \tag{21}$$

$$X=\tfrac{1}{2}(q_1\,\partial/\partial q_1+q_2\,\partial/\partial q_2)+\tfrac{1}{2}(\dot q_1\,\partial/\partial \dot q_1+\dot q_2\,\partial/\partial \dot q_2), \tag{22}$$

$$Y'=-Y, \qquad X'=X.$$

Corollary 2 in this example does not produce non-trivial constants.

We are indebted to Frans Cantrijn for valuable discussions and to Professor Mertens for his interest in our work.

References

Crampin M 1977 *Int. J. Theor. Phys.* **16** 741–54
—— 1980 *Phys. Lett.* **79A** 138–40
González-Gascón F and Rodriguez-Camino E 1980a *Lett. Nuovo Cimento* **27** 363–8
—— 1980b *Lett. Nuovo Cimento* **29** 113–9
—— 1980c *Lett. Nuovo Cimento* **29** 310–4
Hojman S and Harleston H 1981 *J. Math. Phys.* **22** 1414–9
Lutzky M 1978 *J. Phys. A: Math. Gen.* **11** 249–58
—— 1979 *Phys. Lett.* **72A** 86–8
—— 1982a *Phys. Lett.* **87A** 274–6
—— 1982b *J. Phys. A: Math. Gen.* **15** L87–91

Marmo G and Saletan E J 1977 *Nuovo Cimento* **40B** 67–89
Prince G 1983 *Bull. Austr. Math. Soc.* **27** 53–71
Sarlet W and Cantrijn F 1981a *Siam Rev.* **23** 467–94
—— 1981b *J. Phys. A: Math. Gen.* **14** 479–92
Steeb W-H 1982 *Hadronic J.* **5** 1738–47